

Anomalous spreading in a system of coupled Fisher-KPP equations

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Abstract

In this article, we report on the curious phenomena of anomalous spreading in a system of coupled Fisher-KPP equations. When a single parameter is set to zero, the system consists of two uncoupled Fisher-KPP equations which give rise to traveling fronts propagating with the unique, minimal KPP speed. When the coupling parameter is nonzero various behaviors can be observed. Anomalous spreading occurs when one component of the system spreads at a speed significantly faster in the coupled system than it does in isolation, while the speed of the second component remains unchanged. We study these anomalous spreading speeds and show that they arise due to poles of the pointwise Green's function corresponding to the linearization about the unstable homogeneous state. These poles lead to anomalous spreading in the linearized system and come in two varieties – one that persists and leads to anomalous spreading for the nonlinear system and one that does not. We describe the mechanisms leading to these two behaviors and prove that one class of poles are irrelevant as far as nonlinear wavespeed selection is concerned. Finally, we show that the same mechanism can give rise to anomalous spreading even when the slower component does not spread.

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1 Introduction

In this article, we study anomalous spreading speeds in a system of coupled Fisher-KPP equations,

$$\begin{aligned} u_t &= du_{xx} + \alpha(u - u^2) + \beta v(1 - u) \\ v_t &= v_{xx} + (v - v^2). \end{aligned} \tag{1.1}$$

We are interested in the following problem. Consider a positive, compactly supported perturbation of the unstable homogeneous state $(u, v) = (0, 0)$. What is the asymptotic speed of propagation associated to the u component?

Consider first the case when $\beta = 0$ or v is identically zero. Then the equation governing the dynamics of the u component is the scalar Fisher-KPP equation,

$$u_t = du_{xx} + \alpha(u - u^2).$$

This equation has been studied in great detail by a number of authors, see [9, 16, 1, 4] among others. It is well known that compactly supported, positive perturbations of the zero state evolve into a pair of counter propagating fronts. These fronts travel with asymptotic speed $2\sqrt{d\alpha}$. At the same time, observe that the

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v component decouples and its evolution is also given by a Fisher-KPP equation with a selected spreading speed of two. For $\beta > 0$ the evolution of the u component depends on that of the v component. One might conjecture that the selected speed of the u component for the full system will be either 2 or $2\sqrt{d\alpha}$, whichever is larger. This turns out to not be the case, and there are large swaths of the (d, α) parameter plane for which speeds of propagation are observed that exceed both 2 and $2\sqrt{d\alpha}$. See Figure 1 for an illustration of those parameter values leading to faster speeds of propagation in the u component. This phenomena was first observed in [24] and given the label of anomalous spreading. In this paper, we give a complete description of the anomalous spreading that arises in system (1.1) and elucidate the mechanisms leading to these faster than expected speeds of propagation.

A natural starting point for the analysis is the linearization of (1.1) about the homogeneous unstable zero state. The fastest speed of propagation associated to the linearization is called the linear spreading speed and is of interest for many reasons. Practically speaking, the linear spreading speed can be inferred from the singularities of the pointwise Green's function. We delay a precise description of this process until section 2 but we remark that this was first observed within the plasma physics community; see [5, 3] for descriptions of the original work aimed at differentiating between absolute and convective instabilities. From a physical standpoint, one would like to understand the relationship between the linear spreading speed and the nonlinear spreading speed. This is a complicated issue and a complete overarching theory is not known. It has been shown in [18, 19] that a large class of invasion problems are linearly determinate; that is the nonlinear selected speed is exactly the linear spreading speed. These linear determinate fronts are pulled fronts according to the commonly used terminology reviewed for example in [23]. This, of course, does not hold in general and there exists examples for which the selected wavespeed is not the linear one, see for example Nagumo's equation [11]. These nonlinearly determined fronts are referred to as pushed fronts.

Another natural question is whether the linear spreading speed places a lower bound on the speed of propagation for the nonlinear system. Wavespeed selection for general systems of equations is usually described in terms of marginal stability of the invasion front, [6]. The selected front should be marginally stable against compactly supported perturbations, which is to say that perturbations of the traveling front should neither grow nor decay when viewed pointwise in a traveling frame moving with the speed of that front. Since the linear spreading speed describes the spreading speed of perturbations of the unstable homogeneous state, one might imagine that a perturbation placed sufficiently far ahead of the front interface should spread with the linear spreading speed. This seems to suggest that the linear spreading speed should place a lower bound on the speed of propagation for the nonlinear system. This is the case for scalar problems, but for systems this bound no longer holds. This fact was recently shown in an example of a Lotka-Volterra competition model, [13],

$$\begin{aligned} u_t &= \epsilon^2 u_{xx} + (1 - u - a_1 v)u \\ v_t &= v_{xx} + r(1 - a_2 u - v)v. \end{aligned} \tag{1.2}$$

Here $0 < \epsilon \ll 1$, $r > 0$ and $a_1 < 1 < a_2$. The unstable steady state is the point $(u, v) = (0, 1)$. As is the case in (1.1), the linearization about this fixed point has a skew-product structure. On the level of the linearization, the u component decouples and is described by a Fisher-KPP equation that spreads with rate $2\epsilon\sqrt{1 - a_1}$. The linear coupling to the v component occurs as an inhomogeneous term in the v equation and this together with the fast diffusion of the v component leads to $\mathcal{O}(1)$ spreading speeds for this system. A crucial feature leading to this behavior is the unbounded exponential growth in time of the u component in the linearized system. For the non-linear system, no such exponential growth is possible and so the anomalous linear spreading speed does not persist for the nonlinear system. This should be contrasted with the dynamics of system (1.1) as well as the example provided in [24]. In both those cases, anomalous spreading speeds are observed in both the linear and nonlinear systems.

Mathematically, we will show that these anomalous linear spreading speeds arise due to poles of the pointwise Green's function. This is a crucial difference between the scalar and non-scalar cases. Systems of reaction-diffusion equations can support poles of the pointwise Green's function whereas these singularities in the scalar case are always accompanied by a loss of analyticity. Poles of the pointwise Green's function are important because they represent the linear spreading of one component that is induced by the coupling to a component spreading with a different speed. Based upon the results depicted in Figure 1 as well as the examples in [24, 13] it seems that these poles come in two varieties: one that induces faster spreading speeds in the nonlinear system and one that does not. In section 3, we will come to this question and describe

two different mechanisms leading to anomalous linear spreading speeds. One such mechanism persists in the nonlinear system, while the second type does not. We remark that for poles of the pointwise Green's function to exist, the linearization must possess a skew-product structure where some of the components decouple from the others. Problems for which this skew-product structure exist arise in many areas of pattern formation, ecology and chemistry among others, see [2, 15, 17, 20] for recent examples.

One interesting aspect of this invasion process is the range over which the slower v front has an influence over the faster u front. For large values of t , the u component will develop into a traveling front propagating in a medium for which v is essentially zero. In fact, viewed pointwise in a frame traveling with the asymptotic speed of the u component the v component is converging pointwise to zero as $t \rightarrow \infty$. Nonetheless, the v component, while small, is non-zero and has certain spatial decay rates. These decay rates may be adopted by the u component. In contrast to the mechanism that leads to faster linear spreading speeds in (1.2), this mechanism is not destroyed by the introduction of a nonlinearity and faster speeds of propagation can be observed in the nonlinear regime. This picture is somewhat reminiscent of recent work in [12] where a similar long range interaction led to faster observed speeds of propagation. An important distinction exists, however, since the decay rate of the KPP front is not sufficient to determine the selected speed for the u component in (1.1). Instead, the complex process by which the compactly supported initial data evolves towards the traveling front profile plays a role in the nonlinear speed selection. This process appears to be insufficiently understood.

That anomalous spreading exists despite the fact that the v component is converging pointwise to zero in a frame moving with the anomalous spreading speed suggests that this spreading phenomena should not be dependent on the instability of the zero state with respect to perturbations of the v component. In fact, this is the case. For example, suppose that the equation governing the v component in (1.1) was instead

$$v_t = v_{xx} - v.$$

The zero state is stable and any perturbation of that state will relax to zero. However, in the process of this relaxation the v component is non-zero and has certain decay rates as $x \rightarrow \infty$. In an analogous manner to the anomalous spreading for (1.1), the effect of this spatial decay can give rise to faster speeds of propagation in the nonlinear system. We give an explicit example of this in section 4. An important conclusion to be drawn here is that one must use caution when reducing a system of reaction-diffusion equations by setting stable or slowly propagating components to some constant value. The reduced system have have very different dynamics than the original.

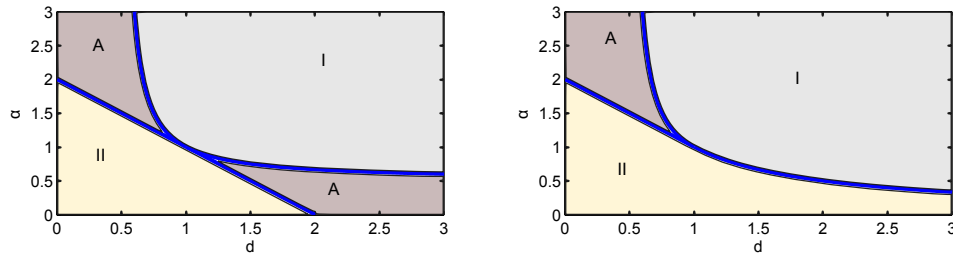


Figure 1: The selected linear spreading speed for the linear (left) and nonlinear (right) systems. Here I denotes a region where the selected speed of the u component is the pulled speed $2\sqrt{da}$, whereas in region II the selected speed is 2. The region A is where anomalous speeds are observed. Note the difference between the linear and nonlinear cases - the anomalous speeds arising due to fast diffusion of the u component do not persist in the nonlinear regime.

The paper is outlined as follows. In section 2 we compute the linear spreading speed using the pointwise Green's function. In section 3, we discuss under what circumstances anomalous spreading in the linearized equation leads to anomalous nonlinear spreading. In section 4, we give an example that shows that this phenomena arises even when the component that induces the anomalous spreading is pointwise stable. In section 5, we contrast numerically observed spreading speeds with the predictions based upon the linearized system. We conclude in section 6 with a short discussion.

2 Spreading speeds in the linearized system

In this section, we compute the linear spreading speed associated to the u component in (1.1). One can think of this speed intuitively as the infimum of all speeds for which a moving observer will outrun the instability. Put another way, the linear spreading speed is the largest speed for which a transition from absolute to convective instability is observed. This transition can be naturally understood in terms of singularities of the pointwise Green's function. After linearizing system (1.1) about the unstable zero state, we will compute these singularities explicitly and calculate the linear spreading speed.

Preliminaries - linear spreading speeds, the pointwise Green's function and pinched double roots of the dispersion relation To begin, we linearize (1.1) about the unstable zero state and consider its evolution in a moving coordinate frame. Let $\xi = x - st$. Then the linearization in this coordinate frame is

$$\begin{aligned} u_t &= du_{\xi\xi} + su_{\xi} + \alpha u + \beta v \\ v_t &= v_{\xi\xi} + sv_{\xi} + v. \end{aligned} \quad (2.1)$$

Since the zero state is unstable, compactly supported perturbations of this state will grow in norm and spread spatially. However, by considering this evolution in a moving frame an important distinction can be made as to whether the instability is absolute or convective in this frame. Let us review the notion of convective versus absolute instabilities. For absolute instabilities, the instability grows both pointwise as well as in norm. When the instability is convective, the instability is transported away from its original location and pointwise decay to zero is eventually achieved despite exponential growth in norm. Mathematically, the question of how to differentiate between convective and absolute instabilities has a long history in the literature, see [5, 3, 14, 21, 23] for the original work within the plasma physics community and subsequent extensions. The key observation is that absolute and convective instabilities can be differentiated according to whether the pointwise Green's function, $G_{\lambda}(\xi - y)$ has singularities in the right half plane.

We review some basic information about the pointwise Green's function $G_{\lambda}(\xi - y)$ for (2.1). We refer the reader to [25] for more information. The pointwise Green's function is computed as follows. After a Laplace transform in time, the linear system (2.1) reduces to a system of second order ordinary differential equations with inhomogeneous right hand side. These equations depend on λ in an analytic fashion and for each λ the corresponding Green's function for this system of equations is denoted $G_{\lambda}(\xi - y)$. That is, we solve

$$\begin{pmatrix} \mathcal{L}_u - \lambda & \beta \\ 0 & \mathcal{L}_v - \lambda \end{pmatrix} \begin{pmatrix} G_{\lambda}^{11} & G_{\lambda}^{12} \\ G_{\lambda}^{21} & G_{\lambda}^{22} \end{pmatrix} = \begin{pmatrix} \delta(y) & 0 \\ 0 & \delta(y) \end{pmatrix}. \quad (2.2)$$

The solution for general initial data can be recovered by convolution. The pointwise Green's function is analytic to the right of the essential spectrum since it is the kernel of the analytic resolvent operator. Pointwise decay is achieved if G_{λ} can be analytically continued into the left half plane. Of course, analyticity properties of $G_{\lambda}(\xi - y)$ depend on the frame of reference, or equivalently on the value of the wavespeed s . For $s = 0$, $G_{\lambda}(\xi - y)$ has a singularity in the right half plane. As s is increased this singularity moves until eventually some value of s_{lin} is reached so that all singularities of $G_{\lambda}(\xi - y)$ remain in the stable half plane for all $s > s_{lin}$. This value of the wavespeed for which the singularity crosses the imaginary axis is the linear spreading speed. That is, we have the following definition.

Definition 1. *The linear spreading speed is*

$$s_{lin} = \sup_{s > 0} \{G_{\lambda}(\xi - y) \text{ has a singularity for some } \lambda \text{ with } \operatorname{Re}(\lambda) > 0\}.$$

In turn, singularities of the pointwise Green's function are related to the eigenvalues of the operators $\mathcal{L}_{u,v} - \lambda$. Denote these eigenvalues $\nu_u^{\pm}(\lambda, s)$ and $\nu_v^{\pm}(\lambda, s)$ where we will henceforth suppress the dependence on λ and s . These eigenvalues are roots of the characteristic polynomials, $d\nu^2 + s\nu + \alpha - \lambda$ and $\nu^2 + s\nu + 1 - \lambda$. The product of these polynomials is the dispersion relation, $d(\nu, \lambda) = d_u(\nu, \lambda)d_v(\nu, \lambda)$, roots of which relate spatial modes $e^{\nu x}$ to their temporal growth rate $e^{\lambda t}$,

$$d(\lambda, \nu) = (d\nu^2 + s\nu + \alpha - \lambda)(\nu^2 + s\nu + 1 - \lambda).$$

To the right of the essential spectrum, the eigenvalues $\nu_{u,v}^+$ lie to the right of the imaginary axis while the stable eigenvalues $\nu_{u,v}^-$ lie to the left of the imaginary axis. Analytic continuation of G_λ depends on the ability to analytically continue the stable and unstable subspaces associated to these eigenvalues into the essential spectrum. Potential issues arise at values of the spectral parameter λ for which an eigenvalue that originated on one side of the imaginary axis aligns with an eigenvalue that originated on the opposite side of the imaginary axis. These are referred to as *pinched double roots* of the dispersion relation, that is, $d(\lambda, \nu)$ has a pinched double root at (λ^*, ν^*) if

$$d(\lambda^*, \nu^*) = 0, \quad \partial_\nu d(\lambda^*, \nu^*) = 0, \quad \text{Re } \nu^\pm(\lambda) \rightarrow \pm\infty \text{ as } \text{Re}(\lambda) \rightarrow \infty.$$

We remark that while pinched double roots may prevent analytic continuation of G_λ , the existence of a pinched double root is not a sufficient condition for the existence of a singularity of G_λ . This is because the pointwise Green's function has an analytic continuation if the stable and unstable subspaces can be continued analytically and pinched double roots do not imply that these subspaces are not analytic. This is easy to observe in our example. If $\beta = 0$, the equations are decoupled and it is clear that a value of λ for which $\nu_u^\pm = \nu_v^\mp$ should not lead to a singularity of G_λ . Nonetheless, for $\beta \neq 0$ these pinched double roots do give rise to singularities of the pointwise Green's function. We will now explicitly compute the pointwise Green's function and show that, in some instances, these pinched double roots induce poles of the pointwise Green's function.

Spreading speeds from the dispersion relation Due to the skew-product nature of the linearization in (2.1), we can compute the roots of the dispersion relation explicitly. We find

$$\begin{aligned} \nu_u^\pm(\lambda, s) &= \frac{-s}{2d} \pm \frac{1}{2d} \sqrt{s^2 - 4d\alpha + 4d\lambda} \\ \nu_v^\pm(\lambda, s) &= \frac{-s}{2} \pm \frac{1}{2} \sqrt{s^2 - 4 + 4\lambda}. \end{aligned} \tag{2.3}$$

Pinched double roots of the full dispersion relation occur for all values of λ , ν and s for which $\nu_{u,v}^+ = \nu_{u,v}^-$. Each pinched double root induces a spreading speed that can be found by selecting s in such a way that the value of λ at the double root satisfies $\text{Re}(\lambda) = 0$. Thus, we set $\lambda = 0$ and solve for s . The spreading speeds associated to the u and v components in isolation are easily found by setting the terms inside the roots in (2.3) to zero. A third spreading speed, the anomalous spreading speed is found by solving for values of s for which $\nu_u^\pm = \nu_v^\mp$ with λ again set equal to zero. Doing this, we find the following three speeds

$$\begin{aligned} s_{u,lin} &:= 2\sqrt{d\alpha} \\ s_{v,lin} &:= 2 \\ s_{anom} &:= \sqrt{\frac{\alpha-1}{1-d}} + \sqrt{\frac{1-d}{\alpha-1}}. \end{aligned}$$

Depending on what parameter values, (d, α) , are under consideration the selected spreading speed of the u component is one of these three speeds. We have the following result.

Theorem 1. *The linear spreading speed for system (2.1) is*

$$s_{lin} = \begin{cases} 2 & \text{for } \alpha < 2-d \\ 2\sqrt{d\alpha} & \text{for } \alpha > \frac{d}{2d-1} \text{ and } (d > \frac{1}{2}) \\ \sqrt{\frac{\alpha-1}{1-d}} + \sqrt{\frac{1-d}{\alpha-1}} & \text{otherwise} \end{cases} \tag{2.4}$$

These regions are plotted in Figure 1. This result can be established by verifying that the pinched double root calculation that leads to the derivation of these speeds actually induces a singularity of the pointwise Green's function. To see this, we must calculate G_λ .

Computing the pointwise Green's function We will now compute G_λ . As is expected, this function will have singularities at the values of λ for which pinched double roots exist. Furthermore, we will observe that these singularities come in two varieties: either G_λ is meromorphic in a neighborhood of the singularity or it fails to be analytic due to a branch point of the dispersion relation.

Recall (2.2). The first component, G_λ^{11} , satisfies the differential equation $(\mathcal{L}_1 - \lambda)G_\lambda^{11} = \delta(y)$, for which it is readily computed that

$$G_\lambda^{11}(\xi - y) = \begin{cases} \frac{1}{\nu_u^- - \nu_u^+} e^{\nu_u^-(\lambda)(\xi - y)} & \text{for } \xi > y \\ \frac{1}{\nu_u^- - \nu_u^+} e^{\nu_u^+(\lambda)(\xi - y)} & \text{for } \xi < y \end{cases}. \quad (2.5)$$

In an analogous fashion, we compute

$$G_\lambda^{22}(\xi - y) = \begin{cases} \frac{1}{\nu_v^- - \nu_v^+} e^{\nu_v^-(\lambda)(\xi - y)} & \text{for } \xi > y \\ \frac{1}{\nu_v^- - \nu_v^+} e^{\nu_v^+(\lambda)(\xi - y)} & \text{for } \xi < y \end{cases}. \quad (2.6)$$

It is easily verified that G_λ^{11} has a singularity and loses analyticity at $\lambda = \alpha - \frac{s^2}{4d}$ while G_λ^{22} does so at $\lambda = 1 - \frac{s^2}{4}$. Selecting $s = s_{u,lin} = 2\sqrt{d}\alpha$ places the singularity of G_λ^{11} on the imaginary axis, while selecting $s = s_{v,lin} = 2$ does the same for the v component. Note that the pinched double roots giving rise to the anomalous spreading speed have no effect on the analyticity of G_λ^{11} or G_λ^{22} . In other words, when $\beta = 0$, the spreading speeds are just these linear spreading speeds and the pinched double root giving rise to the anomalous speed is not dynamically relevant.

However, we are interested in the case of $\beta \neq 0$. In this event, the non-diagonal element G_λ^{12} is non zero and describes the linear behavior of the u component due to coupling with the v component. We will see that the pinched double root $\nu_u^\pm = \nu_v^\mp$ may be relevant here. To compute G_λ^{12} we consider the linear equation

$$(\mathcal{L}_u - \lambda)G_\lambda^{12} + \beta G_\lambda^{22} = 0.$$

Skippping the specifics of the derivation, the general solution is computed on either half line ($\xi > y, \xi < y$) to be

$$G_\lambda^{12}(\xi - y) = \begin{cases} c_+(\lambda) e^{\nu_u^-(\lambda)(\xi - y)} - \frac{\beta}{\nu_u^- - \nu_u^+} \frac{1}{d_u(\nu_v^-, \lambda)} e^{\nu_v^-(\lambda)(\xi - y)} & \text{for } \xi > y \\ c_-(\lambda) e^{\nu_u^+(\lambda)(\xi - y)} - \frac{\beta}{\nu_u^- - \nu_u^+} \frac{1}{d_u(\nu_v^+, \lambda)} e^{\nu_v^+(\lambda)(\xi - y)} & \text{for } \xi < y \end{cases}, \quad (2.7)$$

with the initial conditions enforced at $\xi = y$ so that G_λ^{12} is continuously differentiable. This prescribes the constants

$$\begin{aligned} c_+(\lambda) &= \frac{\beta}{(\nu_u^- - \nu_u^+)(\nu_v^- - \nu_v^+)} \left(\frac{\nu_u^+ - \nu_v^+}{d_u(\nu_v^+, \lambda)} - \frac{\nu_v^- - \nu_v^+}{d_u(\nu_v^-, \lambda)} \right) \\ c_-(\lambda) &= \frac{\beta}{(\nu_u^- - \nu_u^+)(\nu_v^- - \nu_v^+)} \left(\frac{\nu_v^+ - \nu_u^-}{d_u(\nu_v^+, \lambda)} - \frac{\nu_v^- - \nu_u^-}{d_u(\nu_v^-, \lambda)} \right). \end{aligned}$$

We make several immediate observations. First, we see that G_λ^{12} loses analyticity whenever any of the eigenvalues $\nu_{u,v}^\pm(\lambda)$ does. To the right of the branch points $\lambda = \alpha - \frac{s^2}{4d}$ and $\lambda = 1 - \frac{s^2}{4}$ we then have that G_λ^{12} is analytic with the possible exception of those λ values for which $d_u(\nu_v^\pm, \lambda) = 0$, or whenever a spatial eigenvalue for the v subsystem, i.e. ν_v^\pm is equal to a spatial eigenvalue for the u subsystem. At any double root that does not satisfy the pinching condition, the pointwise Green's function has a removable singularity and therefore these points do not pose an obstacle to analytic continuation of G_λ . On the other hand, pinched double roots where $\nu_u^\pm = \nu_v^\mp$ lead to poles of the pointwise Green's function. The location of these poles can be found exactly as was done above, by computing those values of λ for which the roots in (2.3) form pinched double roots. This establishes Theorem 1.

3 Anomalous spreading speeds in the nonlinear system

Whenever $\nu_u^\pm(\lambda) = \nu_v^\mp(\lambda)$ we find that the pointwise Green's function has a pole. However, in numerical simulations we observe that it is only the poles wherein $\nu_u^+ = \nu_v^-$ that give rise to faster observed speeds

of propagation in the nonlinear system, recall figure 1. In this section, we will investigate the physical mechanisms that enforce the existence of these poles and explain why these speeds persist to the nonlinear system only when the double root involved is $\nu_u^+ = \nu_v^-$. In the first case ($\nu_u^+ = \nu_v^-$), we argue that faster spreading speeds are observed due to the alteration of the decay rates of the u component through coupling to the v component. In the second case ($\nu_u^- = \nu_v^+$), we argue that these poles arise due to a combination of the slower growth but faster diffusion associated to the u component relative to the v component. We will also prove that this second type of double root pertains to spreading speeds that are not attainable in the nonlinear system.

The key observation is that the spatial eigenvalues ν_v^\pm can also be interpreted as giving spatial decay rates of some traveling front. These decay rates fall into one of two groups - those with strong decay (ν_v^-) and those with weak decay (ν_v^+). For the nonlinear system, it is known that any front with weak decay is not attainable from compactly supported initial data, see for example [23]. As a result, we will show that any pinched double roots that involve ν_v^+ are not relevant for the nonlinear system. Nonetheless, these double roots lead to poles of the pointwise Green's function and hence anomalous linear spreading. By contrast, the double roots that involve ν_v^- are still relevant for the nonlinear system since any initial condition with this exponential decay rate is dynamically relevant.

Before discussing these two mechanisms in more detail, we will first recall the interpretation of the dispersion relation that relates spatial decay to spreading speeds. Consider the linearized equation,

$$u_t = du_{xx} + \alpha u.$$

As we did when deriving the dispersion relation above, take the ansatz $u(t, x) = e^{\lambda t} e^{\nu x}$ and substitute this into the linear equation. Assuming that $\nu \in \mathbb{R}$ and $\nu < 0$, then we find that $\lambda = d\nu^2 + \alpha$. From this the group velocity,

$$s_g = \frac{\lambda(\nu)}{\nu} = d\nu + \frac{\alpha}{\nu}, \quad (3.1)$$

gives the speed with which this exponential propagates. This relationship can be visualized in (ν, s) space, see Figure 2. For each value of $s > 2\sqrt{d\alpha}$ there are two possible decay rates that propagate with speed s . These values of ν are denoted ν_u^\pm and called spatial eigenvalues as they are exactly the eigenvalues of the fixed point at the origin for the traveling wave equation,

$$du_{\xi\xi} + su_\xi + u - u^2 = 0.$$

Note that the linear spreading speed, $2\sqrt{d\alpha}$ is the value of s where these two branches meet. The upper branch corresponds to traveling front profiles with weak decay. These fronts are stable in an exponentially weighted space, see [22], but are not accessible from compactly supported initial data. On the other hand, if the initial data is not compactly supported but instead has an exponential decay rate belonging to this weak branch then the initial data will evolve into a traveling front moving with the speed prescribed by the group velocity calculation above. For the Fisher-KPP equation, no nonlinear traveling front solutions exist that approach the origin with decay given by the branch ν_u^- . Therefore, initial data with this strong decay evolves towards the KPP traveling front.

The same general picture holds for the v component. Consequently, we see that pinched double roots involving ν_u and ν_v coincide with values in the (ν, s) plane for which the branches ν_u^+ and ν_v^- (or ν_u^- and ν_v^+) intersect. These anomalous spreading speeds are always observed in the linear system. However, when nonlinearities are introduced only the double roots corresponding to $\nu_u^+ = \nu_v^-$ that give rise to anomalous spreading speeds in the nonlinear system. We give physical interpretations of the mechanisms leading to these anomalous linear speeds and use these interpretations to explain why in one case ($\nu_u^+ = \nu_v^-$) they persist in the nonlinear system, while in the other ($\nu_u^- = \nu_v^+$) they do not.

- **Adoption of weak decay by the u component** – In the nonlinear system, the v component is evolving towards a traveling front profile moving with speed 2. Associated to this traveling front profile is a selected decay rate, here $\xi e^{-\xi}$. When this decay rate is shallower than the decay naturally selected by the u front in isolation ($\xi e^{-\sqrt{\frac{d}{\alpha}}\xi}$) then it is possible that the effect of the linear coupling will be sufficient to establish weaker decay rates in the u component than the selected one. These

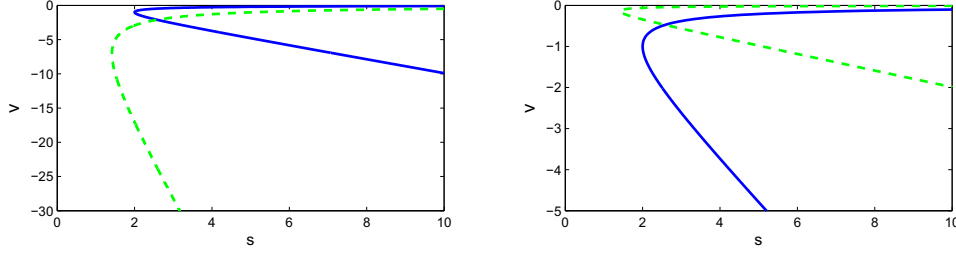


Figure 2: Plots of the group velocity s_g as a function of the spatial decay rate ν . The solid line is this curve for the v component. Note the branch point at the linear spreading speed of 2 with selected spatial decay rate of -1 . The dashed line is the decay rate – group velocity plot for two different values of (λ, nu) . On the left is the curve for $d = 0.1$ and $\alpha = 5$. Note the pinched double root that occurs involving ν_u^+ and ν_v^- . This pinched double root leads to faster observed speeds of propagation in both the linear and nonlinear systems. On the right is the curve for $d = 5$ and $\alpha = 0.1$. Here, the system exhibits anomalous spreading for the linear system but not for the nonlinear system.

weaker decay rates lead to faster than expected speeds of propagation. Furthermore, since it is only the decay rate that matters, this influence can have an effect over a very large spatial scale.

We emphasize that the mechanism leading to faster propagation speeds is more subtle than simply the adoption of the weak decay rates associated to the v front. Faster speeds of propagation are observed only if there exists a double root at $\nu_u^+ = \nu_v^-$. The observed spreading speed is then the linear spreading speed associated to this double root. In general, this speed is slower than the spreading speed associated to weak fronts with decay $\xi e^{-\xi}$. This is due to the rather complicated route that compactly supported initial data take in the convergence to their asymptotic form of a traveling front solution.

- **Weak growth and fast diffusion in the u equation** – There are also double roots and poles of the pointwise Green's function associated to double roots wherein $\nu_n^- = \nu_v^+$. These poles give rise to faster speeds of propagation in the linearized equation, but not in the nonlinear equation – see Theorem 2 below for a proof of this in the nonlinear context.

That anomalous spreading speeds are observed in the linearized equation is apparent from the pointwise Green's function. The physical mechanism leading to this phenomena can be best conceptualized in the limit of large diffusion in the u component. Here, the v component grows and diffuses as usual. These dynamics enter into the u equation as an inhomogeneous source term. The u dynamics adopt the exponential growth of the v component, but the diffusion is much larger for the u component and this leads to faster linear spreading speeds. Note that this was the same mechanism discovered in [13] for the Lotka-Volterra competition model, (1.2).

Observe that this mechanism depends critically on the exponential in time growth of the v component. For the nonlinear system no such growth is possible and therefore this type of anomalous linear spreading does persist for the nonlinear system. This fact is made rigorous in the following Theorem.

Theorem 2. *Consider system (1.1). Suppose that $d > 1$ and $\alpha < \frac{d}{2d-1}$. Then the selected wavespeed of the u component is bounded above by $\max\{2, 2\sqrt{d\alpha}\}$.*

Proof: Let $s > \max\{2, 2\sqrt{d\alpha}\}$. We will construct a front-like super-solution that propagates to the right with speed s , see [8]. To begin, it is known that for any $C_v > 0$,

$$\bar{v}(t, x) = \min\{1, C_v e^{\nu_v^-(0, s)(x-st)}\}$$

is a super-solution for the v dynamics. For every x , the u component converges pointwise as $t \rightarrow \infty$ to the KPP front. That being said, the super-solution above places an upper bound on the spatial decay rate at any fixed time $t > 0$. Thus, the evolution of the u component conserves the steepness of the initial data, see

[7]. We then turn our attention to the u component. Let

$$N(u) = u_t - du_{xx} - \alpha u - \beta v(1 - u) + \alpha u^2.$$

We seek $\bar{u}(t, x)$ so that $N(\bar{u}) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$. Consider

$$\bar{u}(t, x) = \begin{cases} 1 & \text{for } (x - st) < \tau \\ C_u e^{\nu_u^-(x-st)} + C_v \kappa e^{\nu_v^-(x-st)} & \text{for } (x - st) > \tau \end{cases}. \quad (3.2)$$

We will show that we may select τ and $C_u > 0$ so that $\bar{u}(t, x)$ is a super-solution. To begin, one easily verifies that $u = 1$ is a super solution. If τ is taken to be larger than $\tau_{min} := (\nu_v^-)^{-1} \log C_v$ then for $x - st > \tau$ we have that $N(\bar{u})$ is given by

$$N(\bar{u}) = \bar{u}_t - d\bar{u}_{xx} - \alpha\bar{u} - \beta C_v e^{-\nu_v^-(x-st)} + \beta(\bar{v} - v) + \beta v \bar{u} + \alpha \bar{u}^2.$$

Consider

$$\bar{u}_t = d\bar{u}_{xx} + \alpha\bar{u} + \beta C_v e^{-\nu_v^-(x-st)}.$$

This equation has the solution,

$$\tilde{u}(t, x) = C_u e^{\nu_u^-(x-st)} + C_v \kappa e^{\nu_v^-(x-st)},$$

for

$$\kappa = \frac{-\beta}{d(\nu_v^-)^2 + s\nu_v^- + \alpha} < 0.$$

For (d, α) under consideration, we also have that $\nu_v^- < \nu_u^-$. As a result, for any $C_u > 0$ we have that $\tilde{u}(t, x) > 0$ for x sufficiently large. Furthermore, $\tilde{u}(t, x)$ has a unique maximum at

$$\xi_{max} = -\frac{\log(\frac{-C_u \nu_u^-}{C_v \kappa \nu_v^-})}{\nu_u^- - \nu_v^-},$$

for $\xi = x - st$. For $C_u > 0$ and sufficiently large we have that both $\xi_{max} > \tau_{min}$ and the value of $\bar{u}(t, x)$ at this maximum exceeds one. Let τ be chosen so that $\tilde{u}(t, \tau) = 1$. Then $\bar{u}(t, x)$ in (3.2) is continuous. It remains to verify that $N(\bar{u}(t, x)) > 0$ for $x - st > \tau$. On this semi-infinite interval we have,

$$N(\bar{u}(t, x)) = N(\tilde{u}(t, x)) = \beta(\bar{v} - v) + \beta \tilde{u}v + \alpha \tilde{u}^2 > 0,$$

establishing $\bar{u}(t, x)$ as a super-solution.

This construction works for any $s > \max\{2, 2\sqrt{d\alpha}\}$ and therefore establishes that no anomalous spreading speeds are attainable for the nonlinear system in this regime. ■

The relevant case An analogous result for the relevant case is more elusive. Suppose that a pinched double root exists at $\nu_u^+ = \nu_v^-$. One might be tempted to take a dynamical systems approach and seek traveling front solutions. That is, let $V_{KPP}(x - 2t)$ denote the Fisher-KPP traveling front solution. The dynamics of the v component evolve towards this traveling front solution and it therefore might seem natural to study invasion speeds in the inhomogeneous scalar equation,

$$u_t = du_{xx} + \alpha(u - u^2) + \beta V_{KPP}(x - 2t)(1 - u). \quad (3.3)$$

Taking this approach, however, will over-estimate the actual speed of propagation. The effect of the inhomogeneity in (3.3) is to impart the decay rate of the KPP front onto the u component. The selected speed of propagation for (3.3) can then be determined by the group velocity calculation in (3.1). This speed will be larger than the speed selected by compactly supported initial data.

This discrepancy is due to the fact that while for every $\xi = x - 2t$ the v dynamics converge to the traveling front profiled $V_{KPP}(\xi)$ as $t \rightarrow \infty$, the convergence is not uniform. Constructing a sub-solution that shows that the nonlinear anomalous spreading speed is always achieved would require some detailed knowledge concerning the route that the v component takes during its convergence to the traveling front $V_{KPP}(\xi)$. This does not appear to be completely understood in the literature and is beyond the scope of this paper.

4 A stable system inducing anomalous spreading

Viewing the evolution of (2.1) in a frame of reference traveling at the anomalous spreading speed, we observe that the v component is convectively stable. That is, for any fixed $\xi = x - s_{anom}t$ the v component converges pointwise exponentially fast in t to the zero state. Taking this point of view, we expect that the same phenomena may be observed when the v component of the zero state is pointwise stable and not just convectively stable.

We give the following example,

$$\begin{aligned} u_t &= du_{xx} + \alpha(u - u^2) + \beta v(1 - u) \\ v_t &= v_{xx} - \gamma v. \end{aligned} \quad (4.1)$$

We consider compactly supported, positive perturbations of the homogeneous zero state. With this initial data, $v(t, x)$ will decay to zero for all x as $t \rightarrow \infty$. With v converging to zero, one might expect that the u component will then spread with the speed $2\sqrt{d\alpha}$. However, this is not the case as one can observe in Figure 3. The decay rates of $v(t, x)$ as $x \rightarrow \pm\infty$ play an important role in the dynamics despite the fact that the solution is very small and converging to zero. This should be contrasted with the dynamics where v is set identically equal to zero. Here the u component converges, as expected, to the Fisher-KPP front traveling with speed $2\sqrt{d\alpha}$.

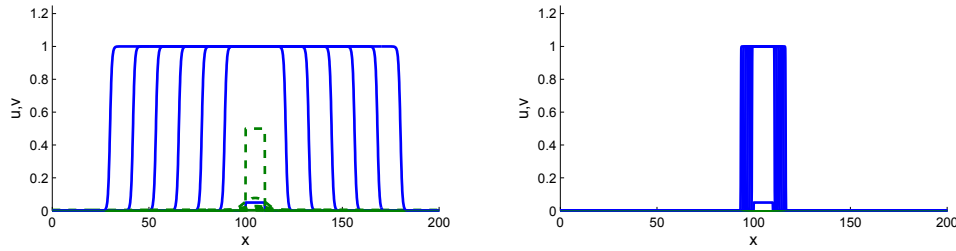


Figure 3: Anomalous spreading for (4.1) with $d = 0.001$, $\alpha = 4$, $\beta = 2$ and $\gamma = 0.1$. The solid line is the u component while the dashed line is the v component. Note that the v component relaxes to zero while the u component spreads. The evolution is shown at times $t = 0, 6, 12, 18, 24, 30, 36$. On the left the evolution is shown for the full system. On the right, the same simulation is run but with identically zero initial conditions for the v component. The spreading speed in the absence of coupling is $2\sqrt{0.004} \approx .1265$ whereas the anomalous speed is approximately 1.9765.

This is completely analogous to the unstable case described above. The linear spreading speed can be determined from the dispersion relation, as before, however the spatial eigenvalues for the v component are now

$$\nu_v^\pm = \frac{-s}{2} \pm \frac{1}{2}\sqrt{s^2 + 4\gamma + 4\lambda}.$$

The evolution of the linearized dynamics are again prescribed by a pointwise Green's function and computing pinched double roots leads to a precise characterization of the linear spreading speed. In particular, we find that poles of the pointwise Green's function arise whenever $\nu_v^- = \nu_u^+$ and for the wavespeed

$$s = \sqrt{\frac{\alpha + \gamma}{1 - d}} - \gamma \sqrt{\frac{1 - d}{\alpha + \gamma}}.$$

This anomalous speed exists for $d < 1/2$ and

$$\alpha > \frac{d\gamma}{1 - 2d}.$$

We remark that for (4.1), if a pinched double root occurs then it is of the appropriate form to lead to faster speeds of propagation in the nonlinear system. If the coupling were in the opposite direction, that is if

the β term were in the v equation rather than the u equation then this pinched root would once again exist and give rise to an anomalous spreading speed in the linear equation, but would not lead to faster speeds of propagation in the nonlinear system. This is what occurs in (1.2) and explains why the fast linear spreading that was observed in that model does not persist to the nonlinear regime, see [13].

5 Numerical simulations

In this final section, we compare numerically derived spreading speeds for the full nonlinear system to the linear spreading speeds predicted in Theorem 1. Recalling our discussion in section 3, we expect to observe anomalous nonlinear spreading only when $d < 1$ and within the parameter regime detailed in Theorem 1. We compute spreading speeds using a Crank-Nicolson type finite difference method. Initial data was chosen to be step functions in both the u component and the v component. Altering the initial location of these step functions relative to one another does not influence the selected wavespeed. Numerically observed wavespeeds for the u component are presented in Figure 4 and Figure 5.

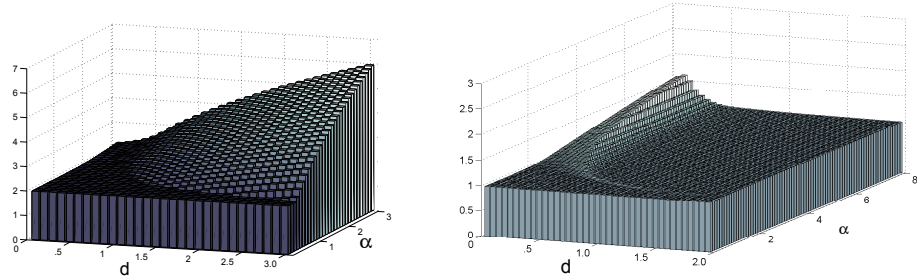


Figure 4: On the left, we plot numerically observed spreading speeds for the u component for $d \in [0, 3]$ and $\alpha \in [0, 3]$. Note that the spreading speed remains two in the linearly anomalous regime for $d > 1$. On the right, we plot the relative error $s_{observed} / \max\{2, 2\sqrt{d\alpha}\}$.

We remark on two qualitative observations. Throughout this article we have considered exclusively the case of $\beta = 2$. It is interesting to note that the particular value of $\beta > 0$ has no influence on the wavespeed selection. Instead, we see that β influences which particular translate of the traveling front that is observed. Furthermore, this translate of the traveling front does not depend on the initial location of the u step function. The selected u front is determined solely by the v component and, after some initial transient, any compactly supported initial data for the u component will converge to this translate.

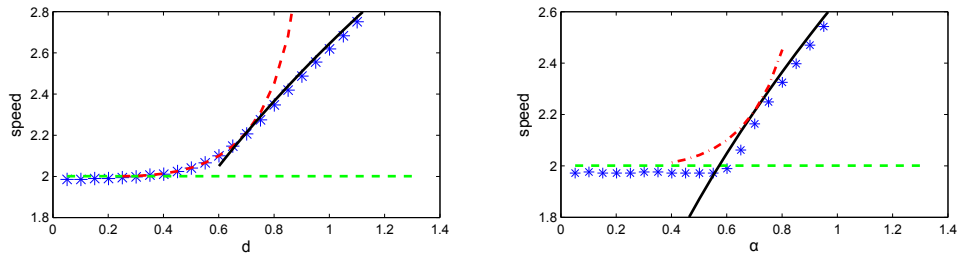


Figure 5: Here we compare numerically derived spreading speeds within two separate sections of the (d, α) parameter plane. On the left, we consider $\alpha = 1.75$ and vary d between 0 and 1.4. The $*$ are numerically derived spreading speeds while the curved dashed line is the anomalous speed, the solid line is the speed $s_{u,lin}$ and the straight dashed line is $s_{v,lin}$. On the right, we do the same comparison, but now with $d = 1.75$ and α varied. As predicted, these parameters lie in the region where there is no anomalous spreading.

6 Discussion

In this article, we have investigated anomalous spreading speeds in a system of coupled Fisher-KPP equations. We conclude with a brief commentary on more general aspects of this phenomena as well as a discussion of unresolved issues.

The fact that the v component completely decouples in the nonlinear system (1.1) is not a prerequisite for anomalous spreading. For example, the nonlinear system

$$\begin{aligned} u_t &= du_{xx} + \alpha(u - u^2) + \beta v(1 - u) \\ v_t &= v_{xx} + \epsilon uv(1 - u) + (v - v^2), \end{aligned}$$

exhibits nonlinear anomalous spreading where the v component travels at speed 2 and the u component travels at the speed selected in the decoupled ($\epsilon = 0$) case. We note that qualitatively similar invasion phenomena has been observed in [2] in the context of an ecological model of the interaction of healthy and diseased red and grey squirrels. This model is much more complicated than (1.1), but numerical simulations reveal behavior reminiscent of anomalous spreading.

One might also be interested in the behavior of the system under some small perturbation that destroys the skew-product nature of the linearization. For example, consider the system

$$\begin{aligned} u_t &= du_{xx} + \alpha(u - u^2) + \beta v(1 - u) \\ v_t &= v_{xx} + \epsilon u(1 - u) + (v - v^2). \end{aligned} \tag{6.1}$$

When $\epsilon = 0$, we recover (1.1). However, for $\epsilon \neq 0$, the linearization about the unstable state no longer has the requisite skew-product structure. In this case, the pinched double root of the dispersion relation that led to a pole of the pointwise Green's function perturbs to a branch point wherein G_λ loses analyticity. These branch points are then dynamically relevant irregardless of whether they corresponded to nonlinearly relevant anomalous spreading speeds in the $\epsilon = 0$ case. That is, for ϵ small and positive the observed spreading speeds for (6.1) are small perturbations of the linear spreading speeds given in Theorem 1 and plotted in the left panel of Figure 1. If $\epsilon < 0$, then positivity is not preserved for (6.1) and solutions diverge. However, if the quadratic nonlinear terms are replaced with cubic ones then for $\epsilon < 0$ the linear spreading speed corresponds to values of λ^* for which $\text{Im}(\lambda^*) \neq 0$. In this regime, the invasion front forms patterns in its wake with long wavelengths. See [10] for a precise example of this phenomena in the context of a phase-field model. Thus, system (1.1) sits on the boundary between systems with invasion fronts that leave patterns in their wake and those that don't.

To conclude, we remark that a rigorous mathematical description of the anomalous invasion process remains an open question. As we pointed out above, the use of a traveling front approach is complicated by several factors. Most glaringly, is that the dynamics of the full system evolve on two separate time scales and therefore no choice of the traveling wave coordinate reduces the problem to an ordinary differential equation. More subtle, is the fact that the observed spreading speed depends on the “transient” evolution of compactly supported initial data towards a traveling front profile.

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